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Identical Particles (Cont'd):

The state vector of a system consisting of N identical bosons (fermions) is symmetric (antisymmetric) under the exchange of any pair of particles. We denote this state vector, which belongs to $V_1 \otimes V_2 \otimes \cdots \otimes V_N$, by $|\psi\rangle_{12\ldots N}$. Thus;

$$|\psi\rangle_{1\ldots i\ldots j\ldots N} = |\psi\rangle_{1\ldots j\ldots i\ldots N} \qquad \text{bosons}$$

$$|\psi\rangle_{1\ldots i\ldots j\ldots N} = -|\psi\rangle_{1\ldots j\ldots i\ldots N} \qquad \text{fermions}$$

The total wavefunction has both the orbital and spin parts:

$$|\psi\rangle_{tot} = \underbrace{|\psi\rangle_{orbit}}_{\in V^{orbit}} \otimes \underbrace{|\psi\rangle_{spin}}_{\in V^{spin}}$$

To elaborate, let us consider a simple toy example with two identical particles in a one-dimensional box of length L. We want to find the wave function of this two particle system for one particle to be in the ground state and the other one in the first excited level.

If the particles are bosons, the wavefunction must be symmetric. Here we only consider orbital part of the wavefunction. Hence, for bosons;

$$|\psi\rangle_b = \frac{1}{\sqrt{2}}\left[|\psi_0\rangle \otimes |\psi_1\rangle + |\psi_1\rangle \otimes |\psi_0\rangle\right]$$

Where $|\psi_0\rangle$ and $|\psi_1\rangle$ are state vectors representing the $n=0$ (ground) and $n=1$ (first excited) states.

In the position space ($|x_1\rangle \otimes |x_2\rangle$ basis) we have:

$$\psi_b(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\,\psi_1(x_2) + \psi_1(x_1)\,\psi_0(x_2)\right]$$

Where: $\left(-\frac{L}{2} \le x \le +\frac{L}{2}\right)$

$$\psi_0(x) = A_0 \cos\left(\frac{\pi x}{L}\right) \quad , \quad \psi_1(x) = A_1 \sin\left(\frac{2\pi x}{L}\right)$$

A_0, A_1 are normalization constants.

On the other hand, for fermions we have:

$$|\psi\rangle_f = \frac{1}{\sqrt{2}}\left[|\psi_0\rangle \otimes |\psi_1\rangle - |\psi_1\rangle \otimes |\psi_0\rangle\right]$$

$$\psi_f(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\,\psi_1(x_2) - \psi_1(x_1)\,\psi_0(x_2)\right]$$

An important point to note is that two fermions cannot be in the same state, because the antisymmetric combination vanishes in this case.

Of course this is true for the total wave function including both the orbital and spin parts. We will discuss this in more detail later on.

Now lets move to the spin part of the wavefunction. We will do it in some special cases as we have not formally introduced spin and angular momentum yet. Spin is a purely quantum mechanical degree of freedom. Particles can have integer spin $(0, 1, \ldots)$ or odd multiples of half spin $(\frac{1}{2}, \frac{3}{2}, \ldots)$.

The former are bosons and the latter are fermions according to the spin-statistics theorem.

The situation is trivial for a spin-0 particles since the Hilbert space that contains the spin degrees of freedom is zero dimensional.

Next is spin-$\frac{1}{2}$ particle. The corresponding Hilbert space V^{spin} is two dimensional. One can consider a basis of spin-up $|+\rangle$ and spin-down $|-\rangle$ states:

$\langle +|+\rangle = \langle -|-\rangle = 1 \qquad \langle +|-\rangle = \langle -|+\rangle = 0$

Now consider two spin-$\frac{1}{2}$ particles (note that the situation is trivial for two or more spin-0 particles).

There are four available spin states to this system:

$|+\rangle \otimes |-\rangle \ , \ |-\rangle \otimes |+\rangle \ , \ |+\rangle \otimes |+\rangle \ , \ |-\rangle \otimes |-\rangle$

For simplicity we show them as:

$|++\rangle \ , \ |+-\rangle \ , \ |-+\rangle \ , \ |--\rangle$

For identical particles, only symmetric and antisymmetric combinations are meaningful. There exist three symmetric states:

$$|\psi\rangle_S : \ |++\rangle, |--\rangle, \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

And one antisymmetric state:

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

Now consider a spin-1 particle, where the Hilbert space V^{spin} is three dimensional. $|+\rangle, |0\rangle, |-\rangle$ are the spin states that form an orthonormal basis:

$$\langle +|+\rangle = \langle 0|0\rangle = \langle -|-\rangle = 1$$

$$\langle +|-\rangle = \langle +|0\rangle = \langle -|+\rangle = \langle -|0\rangle = \langle 0|+\rangle = \langle 0|-\rangle = 0$$

For two spin-1 particles we have 9 spin states:

$$|++\rangle, |00\rangle, |--\rangle, |+-\rangle, |+0\rangle, |0+\rangle, |0-\rangle, |-+\rangle, |-0\rangle$$

For identical particles, only symmetric and antisymmetric combinations are relevant. The symmetric states are:

$|\psi\rangle_S$: $|{+}{+}\rangle, |00\rangle, |{-}{-}\rangle, \frac{1}{\sqrt{2}}\left[|{+}0\rangle + |0{+}\rangle\right], \frac{1}{\sqrt{2}}\left[|{+}{-}\rangle + |{-}{+}\rangle\right], \frac{1}{\sqrt{2}}\left[|0{-}\rangle + |{-}0\rangle\right]$

This makes it 6 symmetric states.

We have three antisymmetric states:

$|\psi\rangle_A$: $\frac{1}{\sqrt{2}}\left[|{+}0\rangle - |0{+}\rangle\right], \frac{1}{\sqrt{2}}\left[|{+}{-}\rangle - |{-}{+}\rangle\right], \frac{1}{\sqrt{2}}\left[|0{-}\rangle - |{-}0\rangle\right]$

Note that for two particles of spin-$\frac{1}{2}$ or spin-1, V^{spin} can be written as the direct sum of two subspaces V_S^{spin} and V_A^{spin} containing the symmetric and antisymmetric states respectively:

$$\text{Spin-}\tfrac{1}{2} \rightarrow 2 \otimes 2 = \underbrace{3}_{V_S^{spin}} \oplus 1 \rightarrow V_A^{spin}$$

$$\text{Spin-}1 \rightarrow 3 \otimes 3 = \underbrace{6}_{V_S^{spin}} \oplus 3 \rightarrow V_A^{spin}$$

However, the situation is different when we have more than two particles. Lets begin with three spin-$\frac{1}{2}$ particles. Now V^{spin} is 8 dimensional. It is easy to see that there exist four symmetric combinations:

$|\Psi\rangle_S$: $|+++\rangle$, $|---\rangle$, $\mathrm{Sym}[|++-\rangle]$, $\mathrm{Sym}[|+--\rangle]$

Here $\mathrm{Sym}[|++-\rangle]$ and $\mathrm{Sym}[|+--\rangle]$ denote symmetric combinations with two + (spin up) and one − (spin down) states, and two − and one + states, respectively.

However, there is no antisymmetric combination. Since each particle has only two spin states, we cannot find a totally antisymmetric combination of spin states for three particles. Note that two of the states must be the same, and hence the antisymmetric combination disappears.

Therefore V_S^{spin} is four dimensional, while V_A^{spin} is zero dimensional.

The other four states in V^{spin} are neither totally symmetric or antisymmetric. They are mixed states that can be symmetric or antisymmetric under the exchange of some particle pairs, but not all.

Similarly, consider three spin-1 particles, V^{spin} is 27 dimensional in this case. There are [redacted] 0 symmetric combinations:

$|\psi\rangle_S$: $|+++\rangle, |000\rangle, |---\rangle$, $\mathrm{Sym}[|++0\rangle]$, $\mathrm{Sym}[|++-\rangle]$, $\mathrm{Sym}[|00+\rangle]$, $\mathrm{Sym}[|00-\rangle]$, $\mathrm{Sym}[|--+\rangle]$, $\mathrm{Sym}[|--0\rangle]$, $\mathrm{Sym}[|+0-\rangle]$

There exists one totally antisymmetric combination:

$$|\psi\rangle_A: \frac{1}{\sqrt{6}}\left[|+0-\rangle - |0+-\rangle + |-+0\rangle - |+-0\rangle + |0-+\rangle - |-0+\rangle\right]$$

Now $V_S^{spin} \oplus V_A^{spin}$ is [redacted] dimensional. The other 16 states are again mixed states.

Now the question arises that whether the mixed states are realized in nature. After all, we have only totally symmetric (antisymmetric) states for a number of identical bosons (fermions).

The point is that we have only considered the spin parts of wavefunctions. The total wavefunction can only be symmetric or antisymmetric. However, the orbital and spin parts can each be a mixed combination, provided that $|\psi\rangle_{tot}$ is totally symmetric or antisymmetric. We will see this explicitly in the context of some examples.